

# A CLASSIFICATION OF POSTCRITICALLY MINIMAL NEWTON MAPS OF ENTIRE FUNCTIONS

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**ABSTRACT.** We obtain a unique, canonical one-to-one correspondence between marked postcritically finite Newton maps of polynomials and postcritically minimal Newton maps of entire functions, which preserves the dynamics and embedding of Julia sets. This bijection is induced by parabolic surgery developed by P. Haïssinsky.

## 1. INTRODUCTION

The Newton map of  $f$  is defined by  $N_f(z) := z - \frac{f(z)}{f'(z)}$ . It is a rational function only when  $f(z) = p(z)e^{q(z)}$  with  $p, q$  polynomials and  $e$  the exponential function, then  $N_{pe^q} = z - \frac{p(z)}{p'(z)+p(z)q'(z)}$ . The finite fixed points of Newton maps are attracting, roots of  $p(z)$ , a point at  $\infty$  is parabolic if  $q(z)$  is not a constant, otherwise it is repelling.

**Definition 1.1** (Postcritically minimal Newton map). A Newton map  $N_{pe^q}$  is called postcritically minimal (PCM) if components of the Fatou set are only of two types: superattracting basins and parabolic basins of  $\infty$ , and the followings hold:

- I. Julia critical orbits and critical orbits in superattracting basins are finite.
- II. critical points in a basin of  $\infty$  are in *minimal* critical orbit relations such that:
  - (a) in every immediate basin of  $\infty$  there exists a single (possibly with higher multiplicity) critical point;
  - (b) all other critical points (if there is any) in a basin of  $\infty$  will land in *minimal iterate* to a critical point in one of immediate basins of  $\infty$  forming a single grant orbit. Moreover, critical grant orbits are minimal: they pass through Fatou components at most once, except at immediate basins of  $\infty$ .

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**Main Theorem 1.2.** *There exists a unique, canonical bijection between marked postcritically finite Newton maps of polynomials and postcritically minimal Newton maps of entire functions, which preserves the dynamics and embedding of Julia sets.*

**Remark 1.3.** In the definition of PCM Newton maps, by minimal critical orbit relations we mean the following. Let a critical point  $c$  in Fatou component  $U$  is captured by a critical point  $c^*$  in an immediate basin  $U^*$ , in particular  $c \neq c^*$ . Let  $k_c$  be a pre-period of  $U$ , i.e.  $f^{\circ k_c}(U) = U^*$  with minimal  $k_c$ . Then we must have  $f^{\circ k_c}(c) = c^*$ . Moreover in the basin of  $U^*$  there exists only a single minimal grant orbit of critical points meaning that the orbit passes through a Fatou component at most once, except immediate basins. This is equivalent to saying that if  $c_1$  and  $c_2$  are critical points such that the orbit of  $c_1$  passes through the component of Fatou set containing the point  $c_2$  then the first passing point should be  $c_2$ .

Recall that a rational function is called postcritically finite (PCF) if its critical orbits are finite; every critical point in the Fatou set (Fatou critical point) eventually terminates at a superattracting periodic point, and every critical point in the Julia set (Julia critical point) eventually terminates at a repelling periodic point. Moreover, in case when a superattracting fixed point (which is also a critical point) captures some other critical point then their critical orbit relation is minimal in the sense that the latter lands to the former without wandering within the immediate basin. Otherwise, its orbit is infinite and never lands to the superattracting fixed point since the dynamics is conjugate to a power map,  $z \mapsto z^k$ . The same landing behavior is required for critical points of PCM Newton maps.

Relaxing a postcritically finiteness condition comes with some cost; postcritical minimality is much weaker than postcritically finiteness. To illustrate how large the family of Newton maps of degree  $d \geq 3$  we are dealing with is, let us fix the degree of polynomials  $p$  and  $q$  to be  $d - n$  and  $n$  ( $1 \leq n \leq d$ ) respectively. Then the parameter space of Newton maps of degree  $d$  (Newton maps for the entire maps  $p(z)e^{q(z)}$ ) is of complex dimension  $d - 2$ . The space of degree  $d$  Newton maps of *polynomials*  $P$  is also of complex dimension  $d - 2$ . But if we write  $d$  as a sum of two non-negative integers,  $d - n$  and  $n \geq 1$ , then it is clear that for every  $d \geq 3$  we have  $d - 1$  distinct “parallel” spaces of complex dimension  $d - 2$  to the space of Newton maps of polynomials.

However, we distinguish and classify all postcritically minimal Newton maps. We shall not build a parallel theory to the successful theory of classification of postcritically finite Newton maps of polynomials. See [LMS] for the full classification of PCF Newton maps of polynomials. Our goal is to transfer this existing knowledge to the new class of functions.

The tool we use is developed by P. Haïssinsky in [Ha98]. The procedure is referred to parabolic surgery. For a Newton map of polynomial (i.e.  $q$  is a constant function), parabolic surgery procedure results to a new rational

function, which turns out to be a Newton map of entire function. This operation defines a map from the space of Newton maps of polynomials to the space of rational Newton maps of entire functions.

For every pair of  $d \geq 3$  and  $1 \leq n \leq d$ , we show that parabolic surgery induces a natural bijection between the space of surgery equivalence classes of marked postcritically finite Newton maps of polynomials, of degree  $d \geq 3$ , and the space of affine conjugacy classes of postcritically minimal Newton maps of entire functions, of degree  $d$ , which are of the form  $\text{id} - \frac{p}{p' + pq'}$  with  $\deg p = d - n$  and  $\deg q = n$ . (Haïssinsky) surgery equivalence class consists of PCF Newton maps with markings of accesses to  $\infty$  in immediate basins of superattracting fixed points, through which parabolic surgeries are applied. PCF Newton maps of polynomials belong to the same equivalence class if the results of parabolic surgeries are affine conjugate, see Definition 4.1 for a more formal definition.

In some sense, we prove stronger result than our Main Theorem. We split the space of PCM Newton maps into disjoint union of “parallel” spaces as in Definition 2.6. The proof has two parts. In part one, we shown that parabolic surgery induces an injective mapping. In the second part, it is shown that the induced mapping is also surjective.

Injectivity of parabolic surgery is given in Theorem 4.2, which shows that results of parabolic surgeries applied to PCF  $N_{p_1}$  and PCF  $N_{p_2}$  with markings are affine conjugate if and only if  $N_{p_1}$  and  $N_{p_2}$  are affine conjugate that sends the markings of  $N_{p_1}$  to the markings of  $N_{p_2}$ .

Surjectivity is given in Theorem 5.1, which states that every PCM Newton map is obtained by a PCF Newton map of polynomial and a parabolic surgery. For this, we use G. Cui’s result on parabolic to hyperbolic surgery to perturb a Newton map of entire function and obtain some Newton map of polynomial with a marking. We then apply parabolic surgery to the latter and obtain a new Newton map of entire function. Finally we show that the Newton map of entire function that we started with and the latter are affine conjugate.

Newton’s method to find roots of a polynomial is classical tool and in recent studies it was shown that it is robust and very efficient even for polynomials of degree over a million, see for more details [SS]. In practical applications, adding an exponential factor comes in disadvantage. When  $\deg q \geq 3$ , M. Haruta in [Har99] showed that the area of every immediate basin of an attracting fixed point is finite. This shows, in particular, that most of the area is taken by the basin of  $\infty$ , where the iterates defined by the Newton’s method applied to  $p(z)e^{q(z)}$  will diverge to  $\infty$ .

Preliminary material is presented and proved in [Ma]. For notions used in holomorphic dynamics please kindly refer to [Mil06].

## 2. DYNAMICAL PROPERTIES OF RATIONAL NEWTON MAPS

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be an entire function (polynomial or transcendental entire function). A meromorphic function given by  $N_f(z) := z - \frac{f(z)}{f'(z)}$  is called the Newton map of  $f(z)$ .

The Newton map  $N_f$  is a rational function if and only if  $f = pe^q$  for some polynomials  $p$  and  $q$ . Let  $m, n \geq 0$  be the degrees of  $p$  and  $q$ , respectively. When  $n = 0$  and  $m \geq 2$ , the point at  $\infty$  is repelling with the multiplier  $\frac{m}{m-1}$ . When  $n = 0$  and  $m = 1$ ,  $N_f$  is constant. If  $n \geq 1$ , the point at  $\infty$  is parabolic with the multiplier  $+1$  and multiplicity  $n + 1 \geq 2$ .

By  $\deg(f, z)$  denote the local degree of a function  $f$  at a point  $z$  and denote  $C_f = \{z \mid \deg(f, z) > 1\}$ . Since for a rational function  $\deg(f, z) > 1$  only when  $z$  is a critical point, thus  $C_f$  consists of critical points of  $f$ . Denote the post-critical set of  $f$  by  $P_f = \bigcup_{n \geq 1} f^{\circ n}(C_f)$ . A rational function  $f$  is called *postcritically finite* (PCF) if  $P_f$  is a finite set. It is called *geometrically finite* if the intersection  $P_f \cap J(f)$  is a finite set.

The *basin of attraction* of an attracting (a parabolic) fixed point  $\xi$  of  $f$  is defined to be, denoted by  $\mathcal{A}(\xi)$ ,

$$\text{int}\{z \in \hat{\mathbb{C}} : \lim_{n \rightarrow \infty} f^{\circ n}(z) = \xi\},$$

the interior of the set of starting points  $z$  that eventually converge to  $\xi$  under iterations of  $f$ . The *immediate basin* of  $\xi$ , denoted by  $\mathcal{A}^\circ(\xi)$ , is the forward invariant connected component of the basin  $\mathcal{A}(\xi)$ . For parabolic fixed points there could be more than one immediate basin.

An immediate basin of a fixed point is simply connected and unbounded for rational Newton maps: in [Prz89], F. Przytycki answering a question posed by A. Manning, see [Man92], proved that for Newton maps of polynomials all immediate basins are simply connected and unbounded. In [MS06], S. Mayer and D. Schleicher extended this result to the case of Newton maps of entire functions. M. Shishikura strengthened these results by proving that *all* components of the Fatou set are simply connected for every rational function with a single weakly repelling fixed point [Shi09], in particular the Julia set is connected for all rational Newton maps of entire functions. Generalizing M. Shishikura's result even further, K. Barański, N. Fagella, X. Jarque, and B. Karpińska in [BFJK] showed that the Julia set is always connected for all (transcendental) Newton maps of entire functions.

A rational Newton map has accesses to  $\infty$  in every immediate basin of attracting fixed point and parabolic fixed point at  $\infty$ . The access in an immediate basin is defined as a homotopy class of curves that start at a point in the immediate basin and ends at  $\infty$ . In attracting domains accesses are repelled from  $\infty$ , i.e backward invariant, while in parabolic domains one access among other repelling accesses gets attracted to  $\infty$ , i.e. forward invariant.

Postcritically minimal Newton maps of entire functions enjoy similar properties as postcritically finite Newton maps of polynomials do. In [Ma], the following result is proved.

**Theorem 2.1** (M, Characterization of PCM Newton maps). *Let  $f$  be a PCM Newton map and  $U$  be a component of the Fatou set of  $f$ , and let  $V = f(U)$ . Then  $U$  contains a unique “center”  $\xi_U$  which is either a critical point or it maps to a point in a superattracting cycle or it maps to a critical point in a parabolic immediate basin of  $\infty$  in finite minimal number of iterations under  $f$ . Moreover, there exist Riemann maps  $\psi_U : U \rightarrow \mathbb{D}$  and  $\psi_V : V \rightarrow \mathbb{D}$  with  $\psi_U(\xi_U) = 0$  and  $\psi_V(\xi_V) = 0$  such that:*

- (a) *if  $U$  is an immediate basin of a parabolic fixed point at  $\infty$  (in this case  $V = U$ ), then the following diagram is commutative*

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ \psi_U \downarrow & & \downarrow \psi_U \\ \mathbb{D} & \xrightarrow{P_k} & \mathbb{D}, \end{array}$$

where  $P_k(z) := \frac{z^k + a}{1 + az^k}$  with  $a = \frac{k-1}{k+1}$ , the parabolic Blaschke product, and  $k-1$  is the multiplicity of a single critical point at  $\xi_U$  in  $U$ ;

- (b) *in all other Fatou components (also including periodic ones), we have the following commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \psi_U \downarrow & & \downarrow \psi_V \\ \mathbb{D} & \xrightarrow{z \mapsto z^k} & \mathbb{D}, \end{array}$$

where  $k-1$  is the multiplicity of a single critical point in  $U$ , if there is no critical point of  $f$  in  $U$  then we let  $k = 1$ .

One of the important objects of this paper is channel diagram of postcritically finite Newton map of *polynomial*. Let the superattracting fixed points of a postcritically finite Newton map  $N_p$  be denoted by  $a_i$ , and their immediate basins by  $\mathcal{A}_i^\circ$  for all  $1 \leq i \leq d$ . Let  $\phi_i : (\mathcal{A}_i^\circ, a_i) \rightarrow (\mathbb{D}, 0)$  be a Böttcher coordinate with the property that  $\phi_i(N_p(z)) = \phi_i^{k_i}(z)$  for each  $z \in \mathbb{D}$ , where  $k_i - 1 \geq 1$  is the multiplicity of  $a_i$  as a critical point of  $N_p$ . The power map  $z \mapsto z^{k_i}$  fixes the  $k_i - 1$  internal rays in  $\mathbb{D}$ . Under  $\phi_i^{-1}$  these rays map to the  $k_i - 1$  pairwise disjoint (except for endpoints) simple curves  $\Gamma_i^1, \Gamma_i^2, \dots, \Gamma_i^{k_i-1} \subset \mathcal{A}_i^\circ$  that connect  $a_i$  to  $\infty$ , are pairwise non-homotopic in  $\mathcal{A}_i^\circ$  and are invariant under  $N_p$  as sets. They represent all accesses to  $\infty$  in  $\mathcal{A}_i^\circ$ .

The union

$$\Delta = \bigcup_{i=1}^d \bigcup_{j=1}^{k_i-1} \overline{\Gamma_i^j}$$

forms a connected graph in  $\hat{\mathbb{C}}$  that is called the *channel diagram* of  $N_p$ . It follows that the channel diagram is forward invariant,  $N_p(\Delta) = \Delta$ . The channel diagram records the mutual locations of the immediate basins of  $N_p$ . Channel diagram of a Newton map tells us all about the possible applications of parabolic surgery. But to apply parabolic surgery we only need single accesses within given immediate basins. So we need to introduce a marking.

**Definition 2.2** (Marked Channel Diagram  $\Delta_n^+$ ). For each  $i \in \{1, \dots, d\}$  we mark *at most one* fixed ray  $\Gamma_i^{j*}$  in the immediate basin of  $a_i$ . If a ray in the immediate basin of  $a_i$  is marked, then we call the basin of  $a_i$  *marked basin*. *Marked channel diagram* is a channel diagram  $\Delta$  with marking, that is an

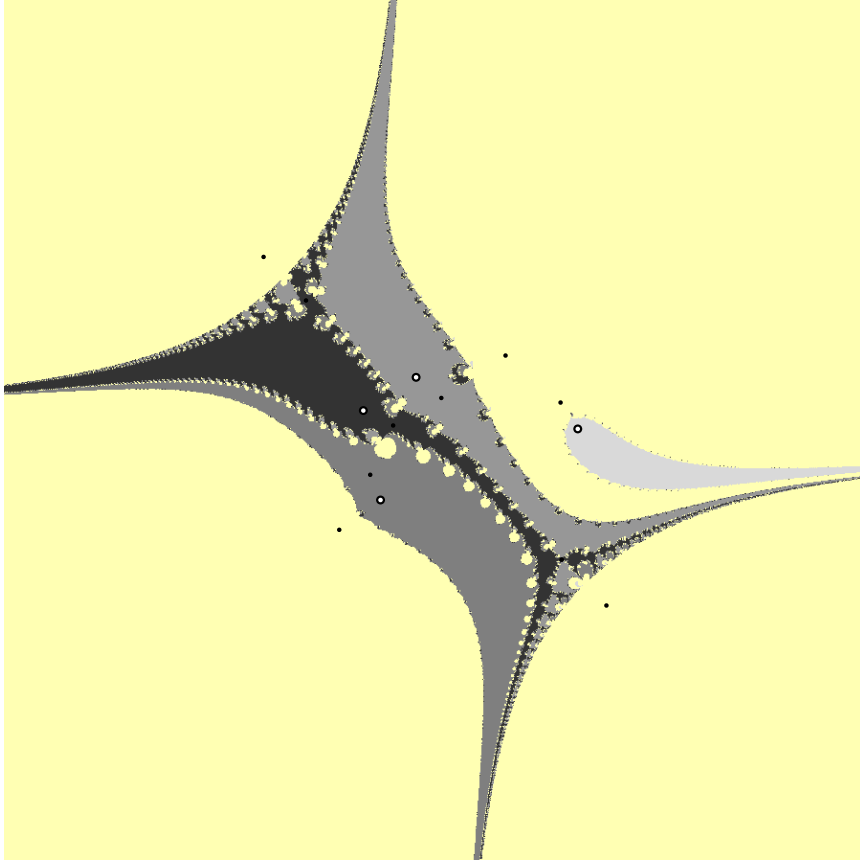


FIGURE 1. The Julia set of the Newton map of degree 8 with basins is depicted. The basin of a parabolic fixed point at  $\infty$  has 4 petals, one of which has two accesses to  $\infty$ , white dots with black circle boundary are fixed points, black dots are non-fixed free critical points.

additional information on which fixed rays are selected/marked. If  $n \leq d$  rays are marked, we denote the marked channel diagram by  $\Delta_n^+$ .

A basin can be marked or unmarked.

**Remark.** Sometimes, we call a channel diagram *unmarked* channel diagram to distinguish it from marked channel diagram.

*Marking* defines a single access among all accesses within a *marked* immediate basin through which parabolic surgery will be performed.

Consider a Newton map  $N_{pe^q}(z) = z - \frac{p(z)}{p'(z) + p(z)q'(z)}$  of degree  $d \geq 3$ , and let  $\deg(q) = n \leq d$ , then the number of distinct roots of  $p$  is  $d - n$ . Notice that the leading coefficient of  $p$  cancels, so we can assume that  $p$  is monic. Similarly, the constant term of  $q$  is also not relevant, since we take the derivative of  $q$ . Any automorphism of  $\hat{\mathbb{C}}$  fixing  $\infty$  is an affine transformation of the form  $z \mapsto az + b$  ( $a \neq 0$ ), which is, in general, a composition of a scaling and a translation. When  $q(z) \not\equiv \text{const.}$ , by scaling, we change the leading coefficient of  $q$  to any nonzero complex number. For instance, we make  $q'$  a monic polynomial. Indeed, a scaling by  $a$  conjugates

$$N_{pe^q}(az)/a = (az - \frac{p(az)}{p'(az) + p(az)q'(az)})/a = z - \frac{p(az)}{ap'(az) + p(az)aq'(az)}.$$

Let  $q'(z) = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \dots$  be the derivative of  $q$ , where  $b_{n-1} \neq 0$  is the leading coefficient of  $q'(z)$ , then we obtain  $aq'(az) = b_{n-1}a^n z^{n-1} + b_{n-2}a^{n-1}z^{n-2} + \dots$ . By a choice of  $a$  such that  $b_{n-1}a^n = 1$ , we make  $q'(z)$  monic. In other words, if we let  $p_a(z) := p(az)$  and  $q_z(z) := q(az)$  then  $N_{pe^q}(az)/a = N_{p_a e^{q_a}}(z)$ . Now we are only left with one more freedom; essentially a translation. By translation, we may further assume that either  $p$  or  $q$  is centered: all roots sum up to zero. If we use the translation by  $b$ , then

$$N_{pe^q}(z + b) - b = z - \frac{p(z + b)}{p'(z + b) + p(z + b)q'(z + b)}.$$

When  $q(z) \equiv \text{const.}$ , by translation we make  $p$  centered; and by scaling we can have  $p(1) = 0$ . We can change the multiplier of an attracting fixed point of a Newton map by a suitable quasiconformal surgery, therefore, we may further assume that all roots of  $p$  are simple: all finite fixed points are superattracting for rational Newton maps of entire functions.

As explained above, let us normalize polynomials  $p$  and  $q$  as follows:

- : **case one;**  $q \equiv \text{const.}$ : we assume that  $p$  is centered and  $p(1) = 0$  (i.e.  $z = 1$  is a root of  $p$ );
- : **case two;**  $q \not\equiv \text{const.}$ : we assume that  $q'$  is monic; moreover, we assume that either  $p$  or  $q$  (the one with the degree at least 2) is centered;
- : furthermore, we assume that  $p$  is monic and has only simple roots (by surgery).

These lead us to define the following main objects of this paper.

**Definition 2.3.** Denote by  $\mathcal{N}(d-n, n)$  the space of degree  $d \geq 3$  Newton maps  $N_{pe^q}$  normalised as above. For instance,  $\mathcal{N}(d) := \mathcal{N}(d, 0)$  is the space of degree  $d \geq 3$  Newton maps of polynomials  $P$ . The polynomials  $P$  are monic and centered, they have a root at  $z = 1$  and all roots are simple.

**Definition 2.4.** Denote by  $\mathcal{N}_{\text{pcf}}(d)$  the space of degree  $d \geq 3$  *postcritically finite* Newton maps for polynomials that are centered, monic and have a root at  $z = 1$ .

**Definition 2.5.** Denote by  $\mathcal{N}_{\text{pcf}}^{+,n}(d)$  the space of all postcritically finite Newton maps in  $\mathcal{N}_{\text{pcf}}(d)$  with all markings  $\Delta_n^+$  (the marked channel diagram with  $n > 0$  markings) at accesses in  $n$  marked immediate basins.

**Definition 2.6 (PCM).** Denote by  $\mathcal{N}_{\text{pcm}}(d-n, n)$  the space of *postcritically minimal* Newton maps in  $\mathcal{N}(d-n, n)$ .

By above arguments and normalization we obtain the following.

**Lemma 2.7.** *Assume functions  $f$  and  $\tilde{f} \in \mathcal{N}(d-n, n)$  are conjugate by an affine map  $\phi$ , i.e.  $\phi \circ f = \tilde{f} \circ \phi$ . Then*

- *if  $n = 0$ , the case of a Newton map of polynomial, then  $\phi(z) = z/a$  where  $a$  is a finite fixed point of  $f$ ;*
- *if  $n \geq 1$  then  $\phi(z) = az$  where  $a^n = 1$ .*

We don't have a true parameter space; some number of maps are conformally conjugate as can be seen in the above lemma. It is also clear now that for every  $n \leq d$  the parameter plane of  $\mathcal{N}(d-n, n)$  is of complex dimension  $d-2$ .

### 3. G. CUI AND PARABOLIC SURGERIES

In [Cui, CT11, CT], G. Cui developed a surgery method, which is called plumbing surgery, to turn parabolic points into hyperbolics: attracting and repelling. Let us state G. Cui's result from [Cui].

**Theorem 3.1 (Cui).** *Suppose that  $g$  is a geometrically finite rational function and  $X$  is a parabolic cycle of  $g$ . Then there exist a continuous family of geometrically finite sub-hyperbolic rational functions  $\{f_t\}$  ( $1 \leq t < \infty$ ) and a continuous family of quasiconformal conjugacies  $\{\phi_t\}$  from  $f_1$  to  $\{f_t\}$ , such that the following conditions hold:*

- (1)  $\{f_t\}$  uniformly converges to  $g$  as  $t \rightarrow \infty$ .
- (2)  $\{\phi_t\}$  uniformly converges to a quotient map  $\phi$  of  $\hat{\mathbb{C}}$  as  $t \rightarrow \infty$  and  $\phi \circ f_1 = g \circ \phi$ , i.e. the following diagram is commutative;

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f_1} & \hat{\mathbb{C}} \\ \phi \downarrow & & \downarrow \phi \\ \hat{\mathbb{C}} & \xrightarrow{g} & \hat{\mathbb{C}}. \end{array}$$



Moreover,  $\phi$  is a homeomorphism from  $J(f_1)$  onto  $J(g)$ .

- (3) For every periodic Fatou domain  $D$  of  $g$ , if  $D$  is a parabolic component associated with  $X$ , then  $\phi^{-1}(D)$  is contained in an attracting domain of  $f_1$  and  $\phi$  is quasiconformal homeomorphism on any domain compactly contained in  $\phi^{-1}(D)$ . Otherwise,  $\phi^{-1}(D)$  is a Fatou domain of  $f_1$  and  $\phi$  is conformal on  $\phi^{-1}(D)$ .

The theorem uses the following notion.

**Definition 3.2** (Quotient map). Let  $h$  be a continuous surjective map on  $\hat{\mathbb{C}}$ . The map  $h$  is called a *quotient* map if  $h^{-1}(y)$  is either a singleton or a full continuum for every point  $y \in \hat{\mathbb{C}}$ , i.e.  $\hat{\mathbb{C}} \setminus h^{-1}(y)$  is a simply connected domain.

**Remark.** Note that  $f_1$  in the above theorem is a sub-hyperbolic geometrically finite function: all of its non-repelling cycles are attracting. The theorem converts all parabolic domains into attracting domains. Since the semi-conjugacy  $\phi$  is conformal in other Fatou components, the multipliers of attracting cycles of  $g$  are preserved. For a postcritically minimal Newton map  $g$ , item (3) of the theorem allows us to conclude that  $f_1$  could be further changed by quasiconformal surgery to a postcritically finite Newton map.

We use the following lemma during construct of a local topological conjugacy between Newton maps at their parabolic fixed points at  $\infty$ , for its proof please refer to [CT, Lemma 3.4.]

**Lemma 3.3.** [CT] Suppose rational maps  $f$  and  $g$  with parabolic fixed points  $z_0$  and  $z_1$  respectively are given. Let  $\phi_0$  be a  $K$ -quasiconformal conjugacy between their attracting flowers. Then for any  $\epsilon > 0$ , there is a local  $(K + \epsilon)$ -quasiconformal conjugacy  $\phi$  between  $f$  and  $g$  such that  $\phi = \phi_0$  on a smaller attracting flower.

We shall use the following fact on extension of quasimetric maps between boundaries of quasidisks, and quasiannuli.

**Proposition 3.4** (Quasiconformal interpolation). [BF14, Proposition 2.30]

- (a) Suppose  $G_1$  and  $G_2$  are quasidisks bounded by  $\gamma_1$  and  $\gamma_2$  and let  $f : \gamma_1 \rightarrow \gamma_2$  be quasimetric. Then  $f$  extends to a quasiconformal map  $\hat{f} : \overline{G_1} \rightarrow \overline{G_2}$ .
- (b) For  $j = 1, 2$ , suppose  $A_j$  are open quasiannuli bounded by the quasicircles  $\gamma_j^i, \gamma_j^o$ . Let  $f^i : \gamma_1^i \rightarrow \gamma_2^i$  and  $f^o : \gamma_1^o \rightarrow \gamma_2^o$  be quasimetric maps between the inner and outer boundaries respectively. Then there exists a quasiconformal map  $f : \overline{A_1} \rightarrow \overline{A_2}$  extending the boundary maps  $f^i$  and  $f^o$ .

In [Ma], we proved the following which is the main tool to construct our bijection between Newton maps.

**Theorem 3.5** (Parabolic surgery for Newton map of polynomial). *Let a postcritically finite Newton map  $N_p$  of degree  $d \geq 3$  with its marked channel diagram  $\Delta_n^+$  be given. Let  $\mathcal{A}(\xi_j)$  be the marked basins of superattracting fixed points  $\xi_j$  for all  $1 \leq j \leq n$ . Then there exist a (David) homeomorphism  $\phi$  and a postcritically minimal Newton map  $N_{\tilde{p}\tilde{e}\tilde{q}}$  of degree  $d$  with  $\deg \tilde{q} = n$  such that:*

- (i)  $\phi(\infty) = \infty$  and  $\phi(\bigcup_{1 \leq j \leq n} \mathcal{A}(\xi_j))$  is the basin of a parabolic fixed point at  $\infty$  of  $N_{\tilde{p}\tilde{e}\tilde{q}}$ ;
- (ii)  $\phi \circ N_p = N_{\tilde{p}\tilde{e}\tilde{q}} \circ \phi$  for all  $z \notin \bigcup_{1 \leq j \leq n} \mathcal{A}(\xi_j)$ ; in particular,  $\phi : J(N_p) \rightarrow J(N_{\tilde{p}\tilde{e}\tilde{q}})$  is a homeomorphism which conjugates  $N_p$  to  $N_{\tilde{p}\tilde{e}\tilde{q}}$ ;
- (iii)  $\phi$  is conformal on the interior of  $\hat{\mathbb{C}} \setminus \bigcup_{1 \leq j \leq n} \mathcal{A}(\xi_j)$ ;
- (iv)  $N_{\tilde{p}\tilde{e}\tilde{q}}$  is postcritically minimal and its attracting accesses of parabolic basins of  $\infty$  correspond to marked accesses of  $\Delta_n^+$ .

The following is a classical result on lifting.

**Lemma 3.6.** *Let  $Y, Z$  and  $W$  be path-connected and locally path-connected Hausdorff spaces with base points  $y \in Y, z \in Z$  and  $w \in W$ . Suppose  $p : W \rightarrow Y$  is an unbranched covering and  $f : Z \rightarrow Y$  is a continuous map such that  $f(z) = y = p(w)$ .*

$$\begin{array}{ccc} Z, z & \xrightarrow{\tilde{f}} & W, w \\ & \searrow f & \downarrow p \\ & & Y, y \end{array}$$

*There exists a continuous lift  $\tilde{f}$  of  $f$  to  $p$  with  $\tilde{f}(z) = w$  for which the above diagram is commutative i.e.  $f = p \circ \tilde{f}$  if and only if*

$$f_*(\pi_1(Z, z)) \subset p_*(\pi_1(W, w)),$$

*where  $\pi_1$  denotes the fundamental group. This lift is unique if it exists.*

#### 4. INJECTIVITY OF PARABOLIC SURGERY

Parabolic surgery defines a mapping from the space of  $n$  marked postcritically finite Newton maps of polynomials (recall that it is denoted by  $\mathcal{N}_{\text{pcf}}^{+,n}(d)$ ) to the space of postcritically minimal Newton maps for  $pe^q$  with  $\deg(q) = n$  (denoted by  $\mathcal{N}_{\text{pcm}}(d - n, n)$ ). We consider all possible applications of parabolic surgery to Newton maps of polynomials of given degree  $d \geq 3$  as one object of study. Different surgeries applied to the same Newton map of a polynomial with different accesses may produce rational functions that are affine conjugate. For the simplest case when  $n = 1$ , we have two ways to apply parabolic surgery to  $\frac{2z^3}{3z^2-1} \in \mathcal{N}_{\text{pcf}}(3)$  along its (two distinct) immediate basins with single accesses to  $\infty$  in each <sup>1</sup>, see Fig. 2 for its Julia set. The resulting Newton map of parabolic surgery has a single basin with

<sup>1</sup>The Newton map  $\frac{2z^3}{3z^2-1}$  is conjugate via  $z \rightarrow \frac{1}{z}$  to the cubic polynomial  $-z^3 + \frac{3}{2}z$ .

single access to  $\infty$ . There exists a single Newton map with that property in  $\mathcal{N}_{\text{pcm}}(2, 1)$ . It is  $z - \frac{z^2+c}{z^2+2z+c}$  for  $c = -\frac{1}{4}$ <sup>2</sup>, see Fig. 3 **Left** for its Julia set. Thus both applications of parabolic surgery produce the same result up to affine conjugacy. The third immediate basin has two distinct accesses.

Similarly, consider applications of parabolic surgery to  $\frac{2z^3}{3z^2-1}$  through its

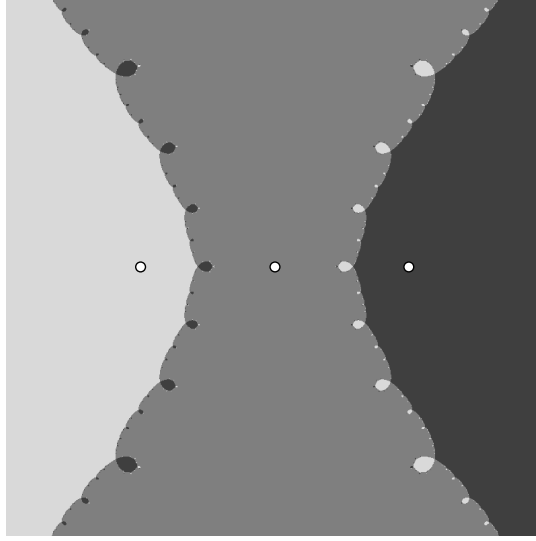


FIGURE 2. The Julia set of  $\frac{2z^3}{3z^2-1}$ , a fixed point at 0 with full invariant basin in gray, the other basins are in light gray and dark gray areas, correspondingly.

third immediate basin. We can perform parabolic surgery in two ways. But the results are again the same function (up to affine conjugacy). It is easy to see that the result is  $z - \frac{z^2+c}{z^2+2z+c}$  for  $c = 2$ <sup>3</sup>. It is a unique Newton map with two accesses in the parabolic immediate basin of  $\infty$ , see Fig. 3 **Right** for its Julia set. We identify these “different” parabolic surgeries if results are Möbius conjugated. It is easy to see that the relation under this identification is an equivalence relation. Let us state it as a definition in the following.

**Definition 4.1** ( $\sim_H$  Haïssinsky equivalence). Let  $F$  and  $G$  be results of applications of parabolic surgery to  $N_{p_1}$  with marking  $\Delta_n^+(N_{p_1})$  and  $N_{p_2}$  with marking  $\Delta_n^+(N_{p_2})$ , both belonging to  $\mathcal{N}_{\text{pcf}}^{+,n}(d)$ , respectively. The two parabolic surgeries are said to be equivalent if there exists an affine map  $M$  such that  $M \circ F = G \circ M$ . Notation  $\sim_H$  is used for equivalent surgeries.

<sup>2</sup>The Newton map  $z - \frac{z^2+c}{z^2+2z+c}$  for  $c = -\frac{1}{4}$  is conjugate via  $z \rightarrow \frac{1}{z} - .5$  to the cubic polynomial  $-z^3 + z^2 + z$ .

<sup>3</sup>The Newton map  $z - \frac{z^2+c}{z^2+2z+c}$  for  $c = 2$  is not a polynomial and it can not be conjugated to a polynomial via Möbius map.

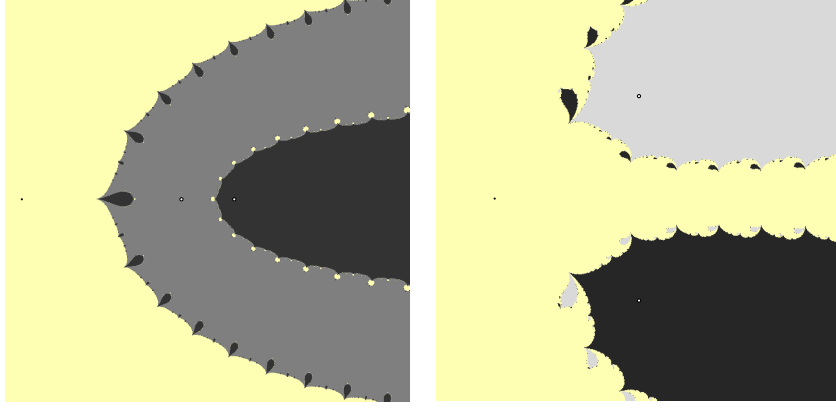


FIGURE 3. **Left:** The Julia set of  $z - \frac{z^2+c}{z^2+2z+c}$  for  $c = -\frac{1}{4}$ , the basin of fixed point  $-0.5$  is in gray area and the basin of other fixed point is in dark gray area, the basin of  $\infty$  is in light yellow area on the left.

**Right:** The Julia set of  $z - \frac{z^2+c}{z^2+2z+c}$  for  $c = 2$ , the basins of fixed points are light gray and dark gray areas, the basin of  $\infty$  has two accesses and is in light yellow area, respectively.

The following theorem characterizes equivalent parabolic surgeries, which states that distinct surgeries produce non-conjugate (distinct) functions unless underlying functions with markings are conjugate themselves and the conjugacy interchanges the markings.

**Theorem 4.2** (Injectivity of parabolic surgery). *Parabolic surgeries applied to  $N_{p_1}$  with marking  $\Delta_n^+(N_{p_1})$  and  $N_{p_2}$  with marking  $\Delta_n'^+(N_{p_2})$ , both belonging to  $\mathcal{N}_{pcf}^{+,n}(d)$ , are equivalent if and only if there exists an affine map  $L$  such that*

- $L \circ N_{p_1} = N_{p_2} \circ L$ ,
- $L(\Delta_n^+(N_{p_1})) = \Delta_n'^+(N_{p_2})$ .

*In other words, the mapping  $\mathcal{F}_n : \mathcal{N}_{pcf}^{+,n}(d) / \sim_H \rightarrow \mathcal{N}_{pcm}(d-n, n)$  induced by parabolic surgery is a unique, canonical, injective mapping, which preserves embedding and the dynamics of Julia sets.*

*Proof.* For one direction: Assume we have an affine map  $L$  such that

- $L \circ N_{p_1} = N_{p_2} \circ L$
- $L(\Delta_n^+(N_{p_1})) = \Delta_n'^+(N_{p_2})$ .

Let us apply parabolic surgery to  $N_{p_1}$  and  $N_{p_2}$  through marked channel diagrams  $\Delta_n^+(N_{p_1})$  and  $\Delta_n'^+(N_{p_2})$  respectively, then the result trivially follows by the construction of parabolic surgery. The converse is the main part of the theorem, which we deal with it now. For the other direction: Let us use simpler notation for the functions involved:  $f = N_{p_1}$ ,  $g = N_{p_2}$ , and let

$F$  and  $G$  be the resulting functions of parabolic surgery to  $f$  with marking  $\Delta_n^+(f)$  and  $g$  with marking  $\Delta_n^+(g)$  respectively. For  $1 \leq j \leq n$ , let us denote by  $\mathcal{A}(\xi_j)$  the marked basins of  $f$ . There exists a homeomorphism  $\phi_f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j) & \xrightarrow{f} & \hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j) \\ \phi_f \downarrow & & \downarrow \phi_f \\ \hat{\mathbb{C}} \setminus \phi_f(\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)) & \xrightarrow{F} & \hat{\mathbb{C}} \setminus \phi_f(\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)), \end{array} \quad \text{D1}$$

where  $\hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)$  is the complement of the union of marked *immediate basins* of  $f$ . Moreover,  $\mathcal{A}_F(\infty) = \phi_f(\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j))$ , which is the parabolic basin of  $\infty$  for  $F$ . As above, for  $1 \leq j \leq n$ , let us denote by  $\mathcal{A}(\chi_j)$  basins of marked superattracting fixed points  $\chi_j$  of  $g$ .

Similarly, there exists a homeomorphism  $\phi_g$  such that the following diagram commutes:

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j) & \xrightarrow{g} & \hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j) \\ \phi_g \downarrow & & \downarrow \phi_g \\ \hat{\mathbb{C}} \setminus \phi_g(\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j)) & \xrightarrow{G} & \hat{\mathbb{C}} \setminus \phi_g(\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j)), \end{array} \quad \text{D2}$$

where  $\hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j)$  is the complement of the union of marked *immediate basins* of  $g$ . Moreover,  $\mathcal{A}_G(\infty) = \phi_g(\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j))$  is the parabolic basin of  $\infty$  for  $G$ .

Assume both surgeries are equivalent:  $F \sim_H G$ , i.e. there exists an affine map  $M$  with the following commutative diagram

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{F} & \hat{\mathbb{C}} \\ M \downarrow & & \downarrow M \\ \hat{\mathbb{C}} & \xrightarrow{G} & \hat{\mathbb{C}}. \end{array} \quad \text{D3}$$

By D3 we obtain  $M(\mathcal{A}_F(\infty)) = \mathcal{A}_G(\infty)$  and  $M(\hat{\mathbb{C}} \setminus \mathcal{A}_F(\infty)) = \hat{\mathbb{C}} \setminus \mathcal{A}_G(\infty)$ , moreover the attracting accesses of  $\mathcal{A}_F(\infty)$  for  $F$  transform via  $M$  to the attracting accesses of  $\mathcal{A}_G(\infty)$  for  $G$ . From diagrams D1, D2 and D3 it follows that

$$\phi_g^{-1} \circ M \circ \phi_f \circ f = g \circ \phi_g^{-1} \circ M \circ \phi_f$$

on  $\hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}(\xi_j)$ . The homeomorphism

$$\psi^1 = \phi_g^{-1} \circ M \circ \phi_f : \hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j) \rightarrow \hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\chi_j)$$

conjugates  $f$  to  $g$  in the complement of the union of marked immediate basins  $\hat{\mathbb{C}} \setminus \cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)$  of  $f$ .

We want to extend  $\psi^1$  to  $\hat{\mathbb{C}}$  as a global conjugacy between  $f$  and  $g$ , and what is missing are the marked immediate basins  $\cup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)$  of  $f$ . To accomplish this we use normalized Riemann maps (Böttcher coordinates)

coming from Theorem 2.1. Let us sort the indices such that  $\mathcal{A}^\circ(\xi_j)$  and their counterparts  $\mathcal{A}^\circ(\chi_j)$  are cyclically ordered at  $\infty$  for  $1 \leq j \leq n$ . Let us pick  $\mathcal{A}^\circ(\xi_j)$  an immediate basin for  $f$ . By Theorem 2.1 there exists a Riemann map  $\psi_{j_f} : (\mathcal{A}^\circ(\xi_j), \xi_j) \rightarrow (\mathbb{D}, 0)$  such that  $\psi_{j_f} \circ f \circ \psi_{j_f}^{-1}(z) = z^{k_j}$ , where  $k_j = \deg(f, \xi_j)$ . We have  $k_j - 1$  choices for  $\psi_{j_f}$ .

Let  $R(t) = \{re^{2\pi it}, 0 \leq r \leq 1\}$  be a *radial line at angle  $t$*  in  $\mathbb{D}$ . We fix some choice of a Riemann map  $\psi_{j_f}$  and define  $R_j(t) = \psi_{j_f}^{-1}(R(t))$ , a *ray of angle  $t$*  in  $\mathcal{A}^\circ(\xi_j)$ . The radial lines  $R(t)$  at angles  $t \in \{0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}\}$  are fixed by  $z \mapsto z^{k_j}$ . Hence, the rays in  $\mathcal{A}^\circ(\xi_j)$  at those angles are fixed by  $f$  define all accesses to  $\infty$  within the immediate basin. Once we label each access, the different choices of  $\psi_{j_f}$  does nothing but cyclically permute (a shift) the labels of accesses. Note that accesses do not depend on a choice of a Riemann map. Let us choose the Riemann map  $\psi_{j_f}$  such that the rays at angles  $0, \frac{1}{k-1}, \dots, \frac{k-2}{k-1}$  in  $\mathcal{A}^\circ(\xi_j)$  are ordered in anti-clockwise direction, 0 ray being the one that was marked. By [TY96] the Julia set of  $f$  is locally connected as it is geometrically finite and its Julia set is connected, so the boundary of every Fatou component is locally connected, hence, every ray lands by Carathéodory's theorem. Also note that every  $f$ -invariant ray lands at  $\infty \in \partial\mathcal{A}^\circ(\xi_j)$ .

We have the same construction for  $g$ : the Riemann maps

$$\phi_{j_g}(\mathcal{A}^\circ(\chi_j), \chi_j) \rightarrow (\mathbb{D}, 0)$$

such that  $\psi_{j_g} \circ g \circ \psi_{j_g}^{-1}(z) = z^{k_j}$ , where  $k_j = \deg(f, \xi_j) = \deg(g, \chi_j)$ . We normalize these Riemann maps of marked immediate basins of  $g$  as well in the same ordering used for  $f$ . We define rays in  $\mathcal{A}^\circ(\chi_j)$ ; similarly every ray for  $g$  also lands.

We construct conjugating maps between corresponding marked immediate basins of  $f$  and  $g$ . Consider the map

$$\psi_j^2 := \phi_{j_g}^{-1} \circ \phi_{j_f} : \mathcal{A}^\circ(\xi_j) \rightarrow \mathcal{A}^\circ(\chi_j)$$

which is conformal. The following diagrams commute;

$$\begin{array}{ccc} \mathcal{A}^\circ(\xi_j) & \xrightarrow{f} & \mathcal{A}^\circ(\xi_j) \\ \psi_{j_f} \downarrow & & \downarrow \psi_{j_f} \\ \mathbb{D} & \xrightarrow{z \mapsto z^{k_j}} & \mathbb{D} \\ \psi_{j_g} \uparrow & & \uparrow \psi_{j_g} \\ \mathcal{A}^\circ(\chi_j) & \xrightarrow{g} & \mathcal{A}^\circ(\chi_j). \end{array}$$

It is now natural to check if both  $\psi^1$  and  $\psi^2$  match up on  $\partial\mathcal{A}^\circ(\xi_j)$ . For this we define an equivalence relation on  $\mathbb{S}^1$  for  $\psi_{j_f}$  (and  $\psi_{j_g}$ ) classes of rays (identified by angles) that land at a common point. Alternatively, since the

inverse to  $\psi_{j_f}$  (correspondingly the inverse to  $\psi_{j_g}$ ) has the continuous extension to the closed unit disk by Carathéodory's Theorem, every equivalence class consists of points of  $\mathbb{S}^1$  that are mapped to the same point under the continuous extension of the inverse of  $\psi_{j_f}$  (correspondingly the continuous extension of the inverse of  $\psi_{j_g}$ ).

All  $f$ -invariant rays land at  $\infty$ , and thus these belong to the same class. All iterated pre-fixed (the image is an invariant ray) rays split into distinct equivalent classes. It is clear that our equivalence relation is generated by the closure of the equivalence relation defined by  $f$ -invariant rays and their iterated preimages. By the normalized Riemann maps,  $\psi_{j_f}$  for  $f$ , and  $\psi_{j_g}$  for  $g$ , we obtain the same equivalence relation for both  $f$  and  $g$ . Indeed, the map  $\psi^1$  sends bijectively the iterated preimages of  $\infty$  in the  $f$  plane to the corresponding iterated preimages of  $\infty$  in the  $g$  plane. Thus  $\psi_j^2$  extends continuously to the closure  $\overline{\mathcal{A}^\circ(\xi_j)}$ . Since  $\infty \in J(f)$  therefore iterated preimages of  $\infty$  are dense in  $\partial\mathcal{A}^\circ(\xi_j)$ , hence for every point  $z \in \partial\mathcal{A}^\circ(\xi_j)$  the equivalent class of rays landing at  $z$  is a limit of classes of rays landing at iterated preimages of  $\infty$  in  $\partial\mathcal{A}^\circ(\xi_j)$ . Moreover, the extension (denote again  $\psi_j^2$ ) coincides with  $\psi^1$  on the iterated preimages of  $\infty$ . By construction the maps  $\psi^1$  and  $\psi_j^2$  agree on a dense subset of their common domains of definition; namely, on the point at  $\infty$  and its iterated preimages in  $\partial\mathcal{A}^\circ(\xi_j)$ . It follows that  $\psi^1$  and  $\psi_j^2$  coincide everywhere on their common domains of definition. Hence the orientation preserving homeomorphism

$$\psi = \begin{cases} \psi^1(z), & z \in \hat{\mathbb{C}} \setminus \bigcup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j) \\ \psi_j^2(z), & z \in \mathcal{A}^\circ(\xi_j), \text{ for } 1 \leq j \leq n, \end{cases}$$

conjugates  $f$  to  $g$  in  $\hat{\mathbb{C}}$ .

Finally, we invoke the rigidity part of Thurston's theorem on characterization of branched coverings [DH93] (actually we need to apply a result from [BCT14], where we add some extra marked points to the postcritical set, for us, it is a point at  $\infty$ ) to degree  $d \geq 3$  functions  $f$  and  $g$  deducing the existence of  $L$ , a conformal conjugacy  $L \circ f = g \circ L$ <sup>4</sup>. Moreover,  $L$  sends the marked fixed critical points of  $\Delta_n^+(f)$  to those of  $\Delta_n^+(g)$ , hence all of the marked channel diagram:  $L(\Delta_n^+(f)) = \Delta_n^+(g)$ .  $\square$

<sup>4</sup>Alternatively, by the proof structure of Chapter 6 of [DH], we can construct the conformal conjugacy by hand by keeping the conformal conjugacy at small disc neighborhoods of superattracting periodic points of  $f$  compactly contained in their immediate basins and interpolating this conformal map to a quasiconformal map  $\phi_0$  of the sphere. Next, we keep taking lifts and obtain a sequence of quasiconformal maps with the same complex dilatation. We only need to require for all  $m > 0$ ,  $\phi_m \circ f = g \circ \phi_{m+1}$  and  $\phi_m = \phi_0$  in a small disc neighborhood of some superattracting fixed point of  $f$  so that we fix a base point from this domain to define the lifts. The sequence  $\{\phi_m\}_{m \geq 0}$  has a convergent subsequence. Let  $\phi$  be its limit. It is clear that  $\phi$  is a conformal map of  $\hat{\mathbb{C}}$  since the domain where  $\phi_m$  are not conformal shrinks to the Julia set of  $f$ . The claim follows since the Julia set of  $f$  has measure zero. In the domain we have  $\phi = \phi_0$ . We have constructed the initial map to satisfy  $\phi_0 \circ f = g \circ \phi_0$  in the domain, hence  $\phi \circ f = g \circ \phi$  by the identity principle of holomorphic functions.

## 5. SURJECTIVITY OF PARABOLIC SURGERY

We use G. Cui's surgery to perturb parabolic fixed points of Newton maps. For a given postcritically minimal Newton map with a parabolic fixed point at  $\infty$ , we change its parabolic domains into attracting, thus producing a sub-hyperbolic rational function, which then is turned to a postcritically finite Newton map of polynomial with its marked accesses to  $\infty$ . The latter is done by a standard surgery: changing multipliers at attracting fixed points and in preimage components of it if there are critical points. Then by parabolic surgery we do the reverse of this process, namely for that postcritically finite Newton map of polynomial we change its repelling fixed point at  $\infty$  into a parabolic fixed point via parabolic surgery, thus obtaining a rational map, which turns out to be a Newton map of entire function. We show that the latter is affine conjugate to the postcritical minimal Newton map of entire function we started with.

**Theorem 5.1** (Surjectivity of parabolic surgery). *For every pair of non-negative integers  $d \geq 3$  and  $1 \leq n \leq d$ , parabolic surgery induces a (natural) surjective mapping  $\mathcal{F}_n$  from the quotient space  $\mathcal{N}_{\text{pcf}}^{+,n}(d)/\sim_H$  onto the space of affine conjugacy classes of Newton maps in  $\mathcal{N}_{\text{pcm}}(d-n, n)$ .*

*Proof.* The proof is involved. For a given function from  $\mathcal{N}_{\text{pcm}}(d-n, n)$  we obtain a new rational function by perturbing a parabolic fixed point at  $\infty$  by Cui plumbing surgery. The obtained function is then converted to a post-critically finite Newton map in  $\mathcal{N}_{\text{pcf}}^{+,n}(d)$  via intermediate surgery. We then apply parabolic surgery to the last Newton map of polynomial to produce a postcritically minimal Newton map. We show that the function we took from  $\mathcal{N}_{\text{pcm}}(d-n, n)$  and the result of parabolic surgery are affine conjugate to each other. Thus, proving that parabolic surgery induces a surjective mapping from the space  $\mathcal{N}_{\text{pcf}}^{+,n}(d)/\sim_H$  to the space of affine conjugacy classes of functions in  $\mathcal{N}_{\text{pcm}}(d-n, n)$ . The proof is split into four major parts, Part A-Part D, as following.

**Part A:** We apply Cui plumbing surgery (Theorem 3.1) to a PCM Newton map  $N_{p_1 e^{q_1}} \in \mathcal{N}_{\text{pcm}}(d-n, n)$  of degree  $d \geq 3$ . We study properties of the resulting rational function  $f_1$  and the quotient map  $\phi$  such that  $\phi \circ f_1 = N_{p_1 e^{q_1}} \circ \phi$ . The injectivity of  $\phi$  is broken only in Fatou components of  $f_1$  that map to parabolic domains of  $N_{p_1 e^{q_1}}$ , in particular, when restricted to  $J(f_1)$ , it is a homeomorphism from  $J(f_1)$  onto  $J(N_{p_1 e^{q_1}})$ . Next, we change  $f_1$  in its attracting basins such that the result of this intermediate surgery produces a post-critically finite Newton map, denote it by  $N_p$ . We have a choice for  $f_1$  but all choices produce the same  $N_p$ , thus we obtain a unique and canonical mapping.

**Part B:** We apply parabolic surgery to  $N_p$  of Part A, with its corresponding marked channel diagram, which is uniquely obtained from  $N_{p_1 e^{q_1}}$ . Denote by  $N_{p_2 e^{q_2}}$  the result of parabolic surgery.



**Part C:** We construct a *topological* conjugacy  $\Psi$  between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$  by cutting parabolic basins where the conjugacy is broken by gluing a conjugacy coming from composing Riemann maps. This part is only needed to make sure that we have correct choices of Riemann maps in parabolic basins, as these are not unique. Alternatively it is possible to skip this part and make this correct choice during the construction of the next part.

**Part D:** Using the topological conjugacy of Part C, which is local conformal on the Fatou set of  $N_{p_1 e^{q_1}}$  and giving up the conjugacy we had, we construct a sequence of quasiconformal homeomorphism that are conformal conjugacies between the Newton maps at petals of parabolic fixed point and in neighborhoods of superattracting basins. The element of the sequence is a lift of the previous and the domain of conjugacy increases eventually filling the whole Fatou set. Finally, by extracting a converging sub-sequence we obtain a *conformal* conjugacy between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$ , finishing the proof of the theorem.

**Part A.** Let a postcritically minimal Newton map  $N_{p_1 e^{q_1}} \in \mathcal{N}_{\text{pcm}}(d-n, n)$  of degree  $d \geq 3$  be given. We invoke Cui plumbing surgery (Theorem 3.1) to deduce a sub-hyperbolic rational function  $f_1$  and a quotient map  $\phi$  such that  $\phi \circ f_1 = N_{p_1 e^{q_1}} \circ \phi$ . Moreover,  $\phi$  is a homeomorphism from  $J(f_1)$  onto  $J(N_{p_1 e^{q_1}})$ . The following diagram is commutative

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f_1} & \hat{\mathbb{C}} \\ \phi \downarrow & & \downarrow \phi \\ \hat{\mathbb{C}} & \xrightarrow{N_{p_1 e^{q_1}}} & \hat{\mathbb{C}}. \end{array}$$

Now we study essential properties of  $f_1$  and  $\phi$ . Without loss of generality we can assume that  $\infty$  is a fixed point of  $f_1$ , after Möbius conjugation if necessary. Then we obtain  $\phi(\infty) = \infty$  since  $\phi(\infty) = N_{p_1 e^{q_1}}(\phi(\infty))$ , and note that  $\infty$  is the only fixed point of  $N_{p_1 e^{q_1}}$  on its Julia set. For the Newton map  $N_{p_1 e^{q_1}}$  the parabolic cycle consists of only a point at  $\infty$ . For every immediate basin  $U$  of  $\infty$  the map  $\phi$  could be obtained as a quasiconformal on any domain compactly contained in  $\phi^{-1}(U)$ , then  $\phi^{-1}$  sends a critical point of  $N_{p_1 e^{q_1}}$  in  $U$  to a critical point of  $f_1$  in  $\phi^{-1}(U)$ . Let  $c \in U$  be a critical point of  $N_{p_1 e^{q_1}}$  in  $U$ . Since  $\phi$  is a homeomorphism restricted to the Julia set, we have  $\deg(f_1, \phi^{-1}(c)) = \deg(N_{p_1 e^{q_1}}, c)$ , thus there is no other critical point of  $f_1$  in  $U$ . Indeed, let  $K$  be a neighborhood of  $\phi^{-1}(c)$  compactly contained in  $\phi^{-1}(U)$ , by the theorem we choose  $\phi$  such that it is quasiconformal on  $K$ , thus  $\phi^{-1}(c)$  is a single point, moreover it is a critical

point of  $f_1$ . The following diagram commutes

$$\begin{array}{ccc} f_1^{-1}(K) & \xrightarrow{f_1} & K \\ \phi \downarrow & & \downarrow \phi \\ \phi(f_1^{-1}(K)) & \xrightarrow{N_{p_1 e^{q_1}}} & \phi(K), \end{array}$$

hence  $\phi$  is quasiconformal on  $f_1^{-1}(K)$ . Induction shows that  $\phi$  is quasiconformal in all of iterated preimages of  $K$ . Now assume  $c_1$  is a critical point of  $N_{p_1 e^{q_1}}$  such that  $N_{p_1 e^{q_1}}^l(c_1) = c \in U$  for a minimal  $l > 0$ . Since  $\phi$  is homeomorphism where the above diagram commutes, it follows that after iteratively applying the conjugacy for iterative preimages of  $K$  we obtain that  $\phi^{-1}(c_1)$  is a critical point of  $f_1$  and  $f_1^l(\phi^{-1}(c_1)) = \phi^{-1}(c)$  for the same minimal  $l > 0$ , moreover since  $\phi$  is a homeomorphism on the Julia set we have  $\deg(f_1, \phi^{-1}(c_1)) = \deg(N_{p_1 e^{q_1}}, c_1)$ . Furthermore, there are no other critical points of  $f_1$  in the Fatou component containing  $\phi^{-1}(c_1)$  than  $\phi^{-1}(c_1)$ .

Similarly, by induction we shall show that  $\phi$  is conformal in every  $\phi^{-1}(U)$ , where  $U$  is a Fatou component of  $N_{p_1 e^{q_1}}$  that is not a parabolic domain. These types of components could only be components of basins of superattracting periodic points (including fixed) of  $N_{p_1 e^{q_1}}$ . If  $U$  is a superattracting immediate basin of  $N_{p_1 e^{q_1}}$  then by Cui plumbing theorem (Theorem 3.1)  $\phi^{-1}(U)$  is an immediate basin of a superattracting periodic point of  $f_1$  and  $\phi$  is conformal in  $\phi^{-1}(U)$ , therefore  $\phi^{-1}$  sends superattracting periodic points of  $N_{p_1 e^{q_1}}$  to those of  $f_1$ . Let  $V$  be a component of  $N_{p_1 e^{q_1}}^{-1}(U)$  other than  $U$ . We have the following commutative diagram

$$\begin{array}{ccc} \phi^{-1}(V) & \xrightarrow{f_1} & \phi^{-1}(U) \\ \phi \downarrow & & \downarrow \phi \\ V & \xrightarrow{N_{p_1 e^{q_1}}} & U, \end{array}$$

hence  $\phi$  is conformal in  $\phi^{-1}(V)$ . By induction,  $\phi$  is conformal in  $\phi^{-1} \circ N_{p_1 e^{q_1}}^{-l}(U)$  for all  $l \geq 1$ . What we have is that for every component of  $F(f_1)$ , that is preserved by the conjugacy  $\phi$ , the critical orbits terminate in finite time.

We have to mention that in all immediate basins of  $f_1$  that are counterparts to the parabolic domains of  $N_{p_1 e^{q_1}}$  we can change the multipliers to zero, see [BF14, Chapter 4.2] and [CG93, Theorem 5.1], compare with [Ma, Lemma 3.8]. By careful checking the process of latter we can achieve that there is a single grant orbit in that basin, then the resulting function is a postcritically finite Newton map, denote it by  $N_p$ . What we have in this process is that the new rational function  $N_p$  and the old  $f_1$  are conjugate except in small neighborhoods of attracting fixed points of  $f_1$ . This intermediate surgery produces a quasiconformal homeomorphism  $\phi_1$  such that

the following diagram is commutative

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus \phi_1^{-1}(A) & \xrightarrow{N_p} & \hat{\mathbb{C}} \setminus \phi_1^{-1}(A) \\ \phi_1 \downarrow & & \downarrow \phi_1 \\ \hat{\mathbb{C}} \setminus A & \xrightarrow{f_1} & \hat{\mathbb{C}} \setminus A, \end{array}$$

where  $A$  is the union of all basins that are *not* affected by the intermediate surgery, moreover  $\phi_1$  is conformal in the interior of  $\hat{\mathbb{C}} \setminus \phi_1^{-1}(A)$ . Let us summarize what we have obtained so far.

- The quotient map  $\phi$ , when restricted to the Julia set of  $f_1$  is a topological conjugacy between the Julia sets of  $f_1$  and  $N_{p_1 e^{q_1}}$ . Moreover,  $\phi$  is a conformal (conjugacy) on the Fatou components of  $f_1$  that counterparts of non-parabolic Fatou domains of  $N_{p_1 e^{q_1}}$ .
- The quasiconformal homeomorphism  $\phi_1$  is a conjugacy between  $f_1$  and  $N_p$  on the complement of the union of disk neighborhoods of (Cui surgery created) attracting fixed points of  $f_1$  and it is conformal in the rest of basins, including all basins of superattracting periodic points of  $f_1$ . Thus,  $\phi \circ \phi_1$ , a quotient map, is a topological conjugacy between the Julia sets of  $N_p$  and  $N_{p_1 e^{q_1}}$ , and it is a conformal map where  $\phi$  is conformal.

Normalize  $N_p$  to make the polynomial  $p$  monic, centered and having a root at 1, so that it belongs to  $\mathcal{N}_{\text{pcf}}(d)$ . We *mark* the basins of  $N_p$  that are created by Cui plumbing surgery. We also need *marked* accesses in every marked basin. By [Ma, Theorem A] every parabolic immediate basin of  $N_{p_1 e^{q_1}}$  has its unique attracting access. Note that, since  $\phi^{-1}$  restricted to the Julia set of  $N_{p_1 e^{q_1}}$  sends homeomorphically boundaries of its parabolic basins to boundaries of attracting basins of  $f_1$ , all (attracting) accesses of former transform to all (marked) accesses of the latter via  $\phi$  (and further via  $\phi_1^{-1}$  to  $N_p$ ). Thus, we have marked accesses of  $N_p$  in its corresponding marked basins that are counterparts to parabolic basins of  $N_{p_1 e^{q_1}}$ .

**Part B.** Now let us denote by  $\mathcal{A}(\xi_j)$  marked basins for  $1 \leq j \leq n$ . We also have marked access in each of  $\mathcal{A}(\xi_j)$ . We apply parabolic surgery (Theorem 3.5) to  $N_p$  through those marked basins and accesses deducing a (David) homeomorphism  $\phi_2$  and a postcritically *minimal* Newton map  $N_{p_2 e^{q_2}}$  such that

- $\phi_2$  is conformal in every Fatou component of  $N_p$  that is not marked,
- $\phi_2 \circ N_p = N_{p_2 e^{q_2}} \circ \phi_2$  for all  $z \notin \bigcup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)$ . i.e. the following diagram commutes

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus \bigcup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j) & \xrightarrow{N_p} & \hat{\mathbb{C}} \setminus \bigcup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j) \\ \phi_2 \downarrow & & \downarrow \phi_2 \\ \hat{\mathbb{C}} \setminus \phi_2(\bigcup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)) & \xrightarrow{N_{p_2 e^{q_2}}} & \hat{\mathbb{C}} \setminus \phi_2(\bigcup_{1 \leq j \leq n} \mathcal{A}^\circ(\xi_j)). \end{array}$$

**Part C.** We shall construct a topological conjugacy between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$  that is conformal in the Fatou set of  $N_{p_1 e^{q_1}}$ . By above constructions it follows that the map  $\Psi = \phi_2 \circ \phi_1^{-1} \circ \phi^{-1}$  is a conjugacy between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$  on the complement of parabolic basins of  $N_{p_1 e^{q_1}}$ . Moreover  $\Psi$  is conformal in the basins of superattracting periodic points (including fixed) of  $N_{p_1 e^{q_1}}$ . We want to extend this conjugacy to parabolic basin  $\mathcal{A}^1(\infty)$  as well. We construct our topological conjugacy by gluing the Riemann maps of corresponding parabolic components. Let  $\mathcal{A}_j^{\circ 1}$  be an immediate basin of parabolic fixed point of  $N_{p_1 e^{q_1}}$  and let  $c_j^1$  be a unique critical point in  $\mathcal{A}_j^{\circ 1}$ , for  $1 \leq j \leq n$ . Note that  $\Psi(\partial \mathcal{A}_j^{\circ 1})$  is a boundary of exactly one parabolic component of  $N_{p_2 e^{q_2}}$  for  $1 \leq j \leq n$ , denote it by  $\mathcal{A}_j^{\circ 2}$ , since it could only be an immediate basin. Let  $c_j^2$  be a unique critical point in  $\mathcal{A}_j^{\circ 2}$ . Let  $\psi_j^1 : \mathcal{A}_j^{\circ 1} \rightarrow \mathbb{D}$  and  $\psi_j^2 : \mathcal{A}_j^{\circ 2} \rightarrow \mathbb{D}$  be the corresponding uniquely defined Riemann maps sending the critical points  $c_j^1$  and  $c_j^2$  to the origin and the fixed point at  $\infty$  to 1 as the parabolic points are accessible, moreover we have  $k_j = \deg(N_{p_1 e^{q_1}}, c_j^1) = \deg(N_{p_2 e^{q_2}}, c_j^2)$ . The following diagrams commute

$$\begin{array}{ccc}
 \mathcal{A}_j^{\circ 1} & \xrightarrow{N_{p_1 e^{q_1}}} & \mathcal{A}_j^{\circ 1} \\
 \psi_j^1 \downarrow & & \downarrow \psi_j^1 \\
 \mathbb{D} & \xrightarrow{P_{k_j}} & \mathbb{D} \\
 \psi_j^2 \uparrow & & \uparrow \psi_j^2 \\
 \mathcal{A}_j^{\circ 2} & \xrightarrow{N_{p_2 e^{q_2}}} & \mathcal{A}_j^{\circ 2},
 \end{array}$$

where  $P_{k_j}(z) = \frac{z^{k_j} + a_j}{1 + a_j z^{k_j}}$  with  $a_j = \frac{k_j - 1}{k_j + 1}$  is a parabolic Blaschke product of  $\mathbb{D}$ . Note that under these normalizations the marked access for both immediate basins are mapped via the Riemann maps to the same access associated to the invariant ray  $(0, 1)$  for  $P_{k_j}$ . For every  $1 \leq j \leq n$ , the composition  $\psi_j^2 \circ (\psi_j^1)^{-1} : \mathcal{A}_j^{\circ 1} \rightarrow \mathcal{A}_j^{\circ 2}$  is a conformal conjugacy on  $\mathcal{A}_j^{\circ 1}$  between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$ .

By Carathéodory's theorem the inverses to both maps  $\psi_j^1$  and  $\psi_j^2$  extend to the boundary of the unit disk. We define an equivalence relation on the unit circle  $\mathbb{S}^1$  induced by extension:  $x \sim y \in \mathbb{S}^1$  if and only if both are mapped to the same point on the boundary by the inverse of  $\psi_j^1$ . Similarly, we define an equivalence relation for the inverse map of  $\psi_j^2$  in other copy of the unit circle  $\mathbb{S}^1$ . We shall show that these two maps define the same equivalence relation on  $\mathbb{S}^1$ . Indeed, we have  $k_j + 1$  fixed points of  $P_{k_j}$ , of which  $k - 2$  are distinct repelling fixed points, and a triple fixed point at 1. In total there are  $k - 1$  invariant accesses, all of them correspond to accesses to  $\infty$  in each of immediate basins  $\mathcal{A}_j^{\circ 1}$  and  $\mathcal{A}_j^{\circ 2}$ . By definition of the equivalence relation, we identify all fixed points  $P_{k_j}$  since they all map

to  $\infty$  under the inverse map. Now we take preimages of a given fixed point of  $P_{k_j}$ . There are  $k_j - 1$  preimages on  $\mathbb{S}^1$  of every fixed point other than the fixed point itself. Similarly, in  $\mathcal{A}_j^{\circ 1}$  the preimages of  $\infty$  by  $N_{p_1 e^{q_1}}$ , since the Newton map is locally injective away from its critical points, the invariant rays/accesses to  $\infty$  have preimages which land at the poles in  $\partial \mathcal{A}_j^{\circ 1}$ , one for each non-homotopic rays/accesses to  $\infty$ . This is transported by the Riemann map  $\psi_j^1$  to the unit disk and we identify preimages of fixed points according to the rules as in  $\mathcal{A}_j^{\circ 1}$ . This gives us  $k_j - 1$  different classes of identifications on  $\mathbb{S}^1$ , one for each corresponding pole other than  $\infty$  of  $N_{p_1 e^{q_1}}$  in  $\partial \mathcal{A}_j^{\circ 1}$ . Continuing this process we identify iterated preimages of all fixed points of  $P_{k_j}$  in  $\mathbb{S}^1$  into equivalence classes coming from iterated preimages of 1 that correspond to the iterated preimages of  $\infty$  on  $\partial \mathcal{A}_j^{\circ 1}$ . Take the closure of this equivalence relation. Since the above diagram commutes we have the same closed equivalent relation on  $\mathbb{S}^1$  for  $\psi_j^1$  and  $\psi_j^2$ .

Thus, the map  $\psi_j^2 \circ (\psi_j^1)^{-1} : \mathcal{A}_j^{\circ 1} \rightarrow \mathcal{A}_j^{\circ 2}$  extends to the boundary as a continuous map and equals to  $\Psi$  on a dense set of points in common domain of definition, namely on  $\infty$  and its iterated preimages. Denote the continuous extension by  $\Psi_j^2$ . Note that  $\Psi_j^2 = \Psi$  on  $\partial \mathcal{A}_j^{\circ 1}$ . The conjugacy is now extended to all of immediate basins of the parabolic fixed point.

Now we extend it to all other components of the parabolic basin  $\mathcal{A}^1(\infty)$ . Let  $U$  be a component of  $N_{p_1 e^{q_1}}^{-1}(\mathcal{A}_j^{\circ 1})$  (iterated preimage of immediate basin) other than  $\mathcal{A}_j^{\circ 1}$ , for some  $1 \leq j \leq n$ . Let  $c_u$  be a unique center of  $U$ , that is a point which maps to a critical point in  $\mathcal{A}_j^{\circ 1}$ , and let  $k = \deg(N_{p_1 e^{q_1}}, c_u)$ . Then  $\Psi(\partial U)$  is the boundary of a unique component of  $N_{p_2 e^{q_2}}(\mathcal{A}_j^{\circ 2})$ , denote the component by  $V$ , and let  $c_v$  denote its unique center. There exist Riemann maps  $\psi_U : U \rightarrow \mathbb{D}$  and  $\psi_V : V \rightarrow \mathbb{D}$  such that  $\psi_U(c_u) = \psi_V(c_v) = 0$  with the following commutative diagrams

$$\begin{array}{ccc}
 U & \xrightarrow{N_{p_1 e^{q_1}}} & \mathcal{A}_j^{\circ 1} \\
 \psi_U \downarrow & & \downarrow \psi_j^1 \\
 \mathbb{D} & \xrightarrow{z \mapsto z^k} & \mathbb{D} \\
 \psi_V \uparrow & & \uparrow \psi_j^2 \\
 V & \xrightarrow{N_{p_2 e^{q_2}}} & \mathcal{A}_j^{\circ 2},
 \end{array}$$

Riemann maps are unique up to post-composing by a rotation of  $k^{\text{th}}$  root of unity. Since we are interested in the composition  $\psi_V^{-1} \circ \psi_U$ , the choice of Riemann maps for both can be restricted to one. Let us fix any choice for  $\psi_U$ . Now we choose the map  $\psi_V$  to be compatible with the dynamics of the Newton maps. Observe that preimages of the invariant rays/accesses (e.g. a marked access, which is associated to the interval  $(0, 1)$ , the zero ray) by  $N_{p_1 e^{q_1}}$  in  $\partial U$  are mapped by  $\psi_U$  to the preimages under  $z \mapsto z^k$  of the

invariant rays landing at fixed points for  $P_{k_j}$  (e.g.  $(0, 1)$ ), since the above diagram is commutative. Note that the map  $z \mapsto z^k$  can not differentiate between different preimages. The map  $\Psi$  that is a homeomorphism from  $\partial U$  onto  $\partial V$  comes in handy. Once  $\psi_U$  is chosen we fix  $\psi_V$  in such a way that those preimages of  $(0, 1)$  by  $z \mapsto z^k$  are pulled back to  $U$  such that they land at the corresponding points dictated by  $\Psi$ . There is only one choice of  $\psi_V$  for doing this. This is compatible with the dynamics of both  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$  on corresponding boundaries of their Fatou components.

We define equivalence relation on  $S^1$  for both  $\psi_U$  and  $\psi_V$  as we did above. These equivalence relations are the same since both agree on a dense set of common points. Hence  $\psi_V^{-1} \circ \psi_U$  extends to the closure of  $U$  and coincides with  $\Psi$  on a dense set of points, thus both are equal on the common domain of definition. This way we extend  $\Psi$  to all (first level) components of preimages of immediate parabolic basins.

We can continue in the same way to extend it to all (iterated preimages) of the parabolic components, since the diagrams are commutative with the same type of model maps  $z \mapsto z^k$ , where  $k$  is a common local degree of Newton maps at centers of components.

Let us summarize what we have proved so far and give (remind) the definition of  $\Psi$ , for which we spent the whole Part C.

$$\Psi = \begin{cases} \phi_2 \circ \phi_1^{-1} \circ \phi^{-1}, & z \in \hat{\mathbb{C}} \setminus \mathcal{A}^1(\infty) \\ \psi_V^{-1} \circ \psi_U, & z \in U, \end{cases}$$

where,  $U$  and  $V$  are component of  $\mathcal{A}^1(\infty)$  and  $\mathcal{A}^2(\infty)$ , correspondingly and all the involved maps in the definition of  $\Psi$  are defined in this and previous parts. Thus,  $\Psi$  is a conjugacy on  $\hat{\mathbb{C}}$  between Newton maps  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$ , it is conformal in every component of Fatou set of  $N_{p_1 e^{q_1}}$ . We still have to show that  $\Psi$  is globally continuous.

**Claim 5.2.** The map  $\Psi$  defined in Part C is a homeomorphism of  $\hat{\mathbb{C}}$ .

*Proof of the Claim.* It suffices to prove continuity of  $\Psi$  on  $J(N_{p_1 e^{q_1}})$ . Let us fix  $\epsilon > 0$ , and a sequence of positive numbers  $\epsilon_k \rightarrow 0$  as  $k \rightarrow \infty$ . Consider a sequence of points  $\{w_s\}_{s \geq 1} \subset \hat{\mathbb{C}}$  such that  $w_s \rightarrow w \in J(N_{p_1 e^{q_1}})$  as  $s \rightarrow \infty$ . We shall prove that

$$(5.1) \quad \Psi(w_s) \rightarrow \Psi(w) \text{ as } s \rightarrow \infty.$$

If for some subsequence of  $w_s$  we have an inclusion  $\{w_{s_k}\}_{k \geq 1} \subset J(N_{p_1 e^{q_1}})$ , then  $\{\Psi(w_{s_k})\}_{k \geq 1} \subset J(N_{p_1 e^{q_1}})$ ; hence, the limit (5.1) holds over the subsequence  $\{w_{s_k}\}_{k \geq 1}$ , since  $\Psi$  is a homeomorphism on the Julia set of  $N_{p_1 e^{q_1}}$ . Moreover, if some subsequence  $\{w_{n_k}\}_{k \geq 1}$  is contained in one of the Fatou components (marked or unmarked) of  $N_{p_1 e^{q_1}}$  (i.e.  $\{w_{n_k}\}_{k \geq 1} \subset U_{N_{p_1 e^{q_1}}}$ ) then along this subsequence the limit (5.1) holds true since the restriction  $\Psi|_{\bar{U}}$  is continuous. Therefore, without loss of generality we assume that  $\{w_s\}_{s \geq 1} \subset F(N_{p_1 e^{q_1}})$  and no subsequence of  $\{w_s\}_{s \geq 1}$  is contained completely in only one of the components of  $F(N_{p_1 e^{q_1}})$ . As a result of this

assumption the sequence  $\{w_s\}_{s \geq 1}$  leaves any given component of  $F(N_{p_1 e^{q_1}})$  in finite time. The Julia set is locally connected so there are only finitely many components of  $F(N_{p_1 e^{q_1}})$  with spherical diameter bigger than any given  $\epsilon > 0$ . Now we fix any  $k$ . Sooner or later the points of  $\{w_s\}_{s \geq 1}$  leave any Fatou component of  $N_{p_1 e^{q_1}}$  with spherical diameter  $\geq \epsilon_k$ . Note that the spherical distance between  $\psi(w_s)$  and  $\psi(w'_s)$  is less than  $\epsilon_k$  for all large enough  $s$ , where  $w'_s$  is any point on the boundary of the component where  $w_s$  is located, in particular,  $w'_s$  is located on  $J(N_{p_1 e^{q_1}})$ . Clearly along the same ideas,  $w'_s \rightarrow w$  as  $s \rightarrow \infty$ . Note that  $w'_s$  converges to the same  $w$ , since  $\Psi$  is continuous on  $J(N_{p_1 e^{q_1}})$ . The claim is now proved.  $\square$

**Part D**<sup>5</sup>. Using the topological conjugacy of Part C, which is, in particular, a conformal map at the petals and superattracting cycles of  $N_{p_1 e^{q_1}}$ , by applying interpolation technique several times we construct a set of quasiconformal homeomorphisms of  $\hat{\mathbb{C}}$ , which is denoted by  $\{\Psi_1, \dots, \Psi_k\}$ , where  $k$  is a total number of superattracting periodic points of  $N_{p_1 e^{q_1}}$ .

Next we work with  $\Psi_k$ , from the previous step, and construct, by taking lifts of a local conjugation,  $\{\psi_m\}_{m \geq 0}$  a sequence of quasiconformal homeomorphisms of  $\hat{\mathbb{C}}$  with bounded complex dilatation. Finally, by extracting a converging sub-sequence of the latter we obtain a *conformal* conjugacy between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$ , finishing the proof of the theorem. We divide the dynamical plane of  $N_{p_1 e^{q_1}}$  into two parts: some Jordan neighborhood of infinity and the complement of it, which is bounded.

We use  $\Psi$  as an initial partial conjugacy between petals at  $\infty$  of  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$ . Note that, in particular,  $\Psi$  is a conformal conjugacy restricted on immediate basins of  $\infty$ . Let us fix an  $\epsilon = 1$  (the exact value of  $\epsilon$  is not relevant). Since  $\Psi$  is conformal in petal, thus 1-quasiconformal homeomorphism, by Lemma 3.3 we obtain a  $1 + \epsilon = 2$ -quasiconformal homeomorphism  $\phi$  defined locally at  $\infty$ , that is a conjugacy between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$  such that  $\phi = \Psi$  on a smaller attracting flower bounded by curves  $l_1, \dots, l_n$ , see Fig. 4. In the figure,  $l_1, \dots, l_n$  denote the boundaries of small petals of  $\infty$ . Extend the conjugacy to big petals to include critical points, denote the extended conjugacy and petal boundaries with previously used notations. The extension is possible thanks to conformality of  $\Psi$ , thus it still conjugates the Newton maps. We fix some quasicircle  $L_1$  in the domain of definition of  $\phi$  such that  $L_1$  separates all superattracting periodic points (including fixed points) of  $N_{p_1 e^{q_1}}$  from the big flower, including critical points and their orbits, where we had the equality  $\phi = \Psi$ . Denote by  $L_1^+$  and  $L_1^-$  the unbounded and bounded components of the complement of  $L_1$  respectively. Consider  $L_2 = \phi(L_1)$  the corresponding quasicircle in the dynamical plane of  $N_{p_2 e^{q_2}}$ .

<sup>5</sup>Note that the topological conjugacy  $\Psi$  of Part C is not necessarily a  $c$ -equivalence between  $N_{p_1 e^{q_1}}$  and  $N_{p_2 e^{q_2}}$  according to the generalization of Thurston's topological characterization of postcritically finite covering maps to the setting of geometrically finite covering maps with parabolic cycles (please refer to [CT] for details).

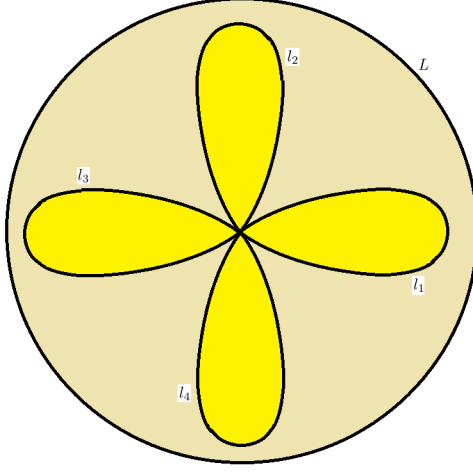


FIGURE 4. A schematic picture of a neighborhood of  $\infty$  with a flower in yellow, for  $n = 4$ .

Moreover,  $L_2$  separates the attracting flower from all superattracting periodic points of  $N_{p_2e^{q_2}}$ . Similarly, denote by  $L_2^+$  the unbounded component of the complement of  $L_2$ , and by  $L_2^-$  the bounded component.

We shall extend  $\phi$  to the bounded domain  $L_1^-$  as a quasiconformal homeomorphism that is conformal on disk neighborhoods of superattracting cycles of  $N_{p_1e^{q_1}}$ . Moreover, it will be equal to the map  $\Psi$  defined above, thus a conformal conjugacy between  $N_{p_1e^{q_1}}$  and  $N_{p_2e^{q_2}}$  on neighborhoods of superattracting periodic points (including fixed).

In case when there exist no superattracting periodic (including fixed) points of  $N_{p_1e^{q_1}}$  we extend  $\phi$  using Proposition 3.4 part (a) to  $L_1^-$ . In case when there exist superattracting periodic points or fixed points of  $N_{p_1e^{q_1}}$ , we extend  $\phi$  using Proposition 3.4 part (b) to  $L_1^-$  sequentially in small disks about all periodic points specified below. Let  $C_1, C_2, \dots, C_k$  denote a list of disjoint simple closed analytic curves contained in  $L_1^-$ , one for each element of superattracting cycles (the critical point and its orbit) that bound an element in its immediate basin for  $N_{p_1e^{q_1}}$ . For every  $i \leq k$  let  $\Omega_i^1$  be the closed disk bounded by  $C_i$ , then  $\Omega_i^1 \Subset L_1^-$ . Note that, the images  $\Psi(\Omega_1^1), \Psi(\Omega_2^1), \dots, \Psi(\Omega_k^1)$  are closed disks in  $L_2^-$  bounded by analytic curves  $\Psi(C_1), \Psi(C_2), \dots, \Psi(C_k)$ , each of which surrounds the corresponding superattracting periodic point (including fixed points) of  $N_{p_2e^{q_2}}$  in its immediate basin.

We are in position to apply Proposition 3.4 part (b). First, consider a quasiannulus with the internal boundary  $C_1$  and the external boundary  $L^1$ . Interpolate inner and outer maps  $\Psi|_{C_1}$  and  $\Psi|_{L^1}$  using Theorem 3.4 part (b) to produce a quasiconformal homeomorphism of  $\hat{\mathbb{C}}$ , denote it by



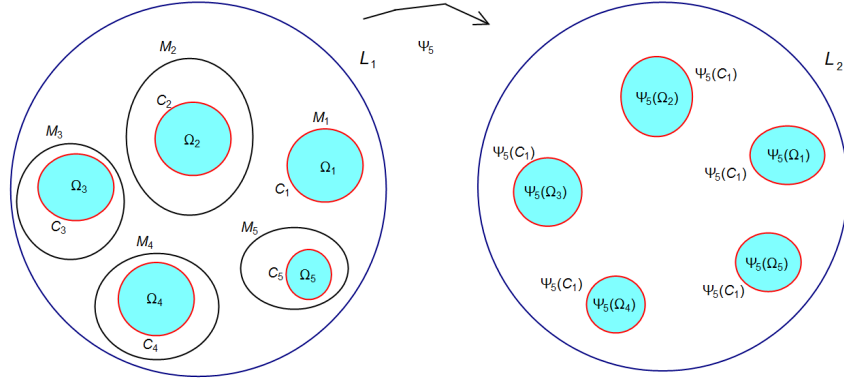


FIGURE 5. A schematic illustration of the construction of the interpolation. Left: The analytic disks in cyan are  $\Omega_i^1$  in the  $N_{p_1 e^{q_1}}$  plane. Right: The corresponding image for the  $N_{p_2 e^{q_2}}$  plane.

$\Psi_1$ . Second, we continue the application of Proposition 3.4 part (b) with the next analytic curve  $C_2$  and the map  $\Psi_1$ , which is obtained in the first step. We need to specify the boundary maps, too. One way to define the boundaries is shrinking the curve  $C_2$ , while keeping the center unchanged, which is a superattracting periodic point (or fixed) of  $N_{p_1 e^{q_1}}$ . By shrinking we mean that we take an analytic curve  $\tilde{C}_2$  within  $\Omega_2^1$ . Another way is taking some quasicircle located within  $L_1^-$ , denote it by  $M_2$ , which bounds the curve  $C_2$  and separates it from  $C_1$ . In the latter case, we have  $\Psi_1|_{M_2}$  and  $\Psi_1|_{C_2}$  as external and internal maps, respectively. Interpolation gives us a quasiconformal homeomorphism of  $\hat{\mathbb{C}}$ , denote this map by  $\Psi_2$ . Note that,  $\Psi_2$  is conformal on the union of  $\Omega_1^1$  and  $\Omega_2^1$ . Finally, we take some quasicircle located within  $L_1^-$ , denote it by  $M_k$ , that bounds the curve  $C_k$  and separates it from all other curves  $C_1, C_2, \dots, C_{k-1}$ . We consider  $\Psi_{k-1}|_{M_k}$  and  $\Psi_{k-1}|_{C_k}$  as external and internal maps respectively for the next interpolation. We obtain a quasiconformal homeomorphism of  $\hat{\mathbb{C}}$ , denote it by  $\Psi_k$ . Note that  $\Psi_k$  is conformal on all of  $\Omega_i^1$  for  $i \leq k$ .

To ease the notations let us denote the last interpolating map by  $\psi_0$  i.e.  $\psi_0 = \Psi_k$  and denote by  $\Omega^1$  the union of  $\Omega_i^1$  for  $1 \leq i \leq k$  and the open parabolic flower bounded by  $l_1, \dots, l_n$  and let  $\psi_0(\Omega^1) = \Omega^2$ . By construction  $\psi_0 = \Psi$  and  $\psi_0 \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_0$  on  $\Omega^1$ . The following diagram commutes

$$\begin{array}{ccc} \Omega^1 & \xrightarrow{N_{p_1 e^{q_1}}} & N_{p_1 e^{q_1}}(\Omega^1) \\ \phi_0 \downarrow & & \downarrow \phi_0 \\ \Omega^2 & \xrightarrow{N_{p_2 e^{q_2}}} & N_{p_2 e^{q_2}}(\Omega^1). \end{array}$$

**Lifting to obtain a sequence of quasiconformal maps.** Recall that  $C_f$  denotes the set of critical points of  $f$ . Let us define sets:  $V^i = N_{p_i e^{q_i}}(C_{N_{p_i e^{q_i}}})$  the set of critical values of  $N_{p_i e^{q_i}}$ , and for  $i \in \{1, 2\}$  let  $T^i = N_{p_i e^{q_i}}^{-1}(V^i)$  be the full preimage of  $C_{N_{p_i e^{q_i}}}$  under  $N_{p_i e^{q_i}}$ .

We have unbranched covering maps  $N_{p_1 e^{q_1}} : \hat{\mathbb{C}} \setminus T^1 \rightarrow \hat{\mathbb{C}} \setminus V^1$  for  $i \in \{1, 2\}$ . Note that if  $\phi_0(V^1) \neq V^2$  then we include the sets  $\phi_0^{-1}(V^2)$  and  $\phi_0^{-1}(V^1)$  to  $V^1$  and  $V^2$  respectively. We define corresponding  $T^i$ 's. The maps  $\psi_0 \circ N_{p_1 e^{q_1}} : \hat{\mathbb{C}} \setminus T^1 \rightarrow \hat{\mathbb{C}} \setminus V^2$  and  $\psi_0^{-1} \circ N_{p_2 e^{q_2}} : \hat{\mathbb{C}} \setminus T^2 \rightarrow \hat{\mathbb{C}} \setminus V^1$  are unbranched covering maps. Let  $\Omega^0$  be a component of  $\Omega^1$ . The Newton map has at least one petal, so as  $\Omega^0$  we take a petal of  $\infty$ . Let us fix a base point  $x_0 \in \Omega^0 \setminus O(C_{N_{p_2 e^{q_2}}})$  for the domain  $\hat{\mathbb{C}} \setminus V^2$ , where  $O(C_{N_{p_2 e^{q_2}}})$  denotes the union of grand orbits of critical points of  $N_{p_2 e^{q_2}}$ . Note that we have  $V^2 \cup T^2 \subset O(C_{N_{p_2 e^{q_2}}})$ . Actually more is true; the grand total orbit of critical points  $O(C_{N_{p_i e^{q_i}}})$  is generated by  $V^i$  and also by  $T^i$  for  $i \in \{1, 2\}$ . Note that  $N_{p_2 e^{q_2}}^{-1}(\Omega^0)$  has many components in the immediate basin associated to the petal  $\Omega^0$ , since  $\Omega^0 \subset N_{p_2 e^{q_2}}^{-1}(\Omega^0)$ . As a base point for the domain  $\hat{\mathbb{C}} \setminus T^2$  let us fix a preimage, denoted by  $y_0$ ,  $N_{p_2 e^{q_2}}^{-1}(x_0)$  in  $N_{p_2 e^{q_2}}^{-1}(\Omega^0)$ . Preimages  $\psi_0^{-1}(x_0)$  and  $\psi_0^{-1}(y_0)$  are base points for domains associated to  $N_{p_1 e^{q_1}}$ , as  $\psi_0$  is bijection. The map  $\psi$  is a homeomorphism therefore the induced maps on fundamental groups of involved domains are isomorphisms. We can invoke Lemma 3.6, the unique lift  $\psi_1$  of  $\psi_0 \circ N_{p_1 e^{q_1}}$  is a map from  $\hat{\mathbb{C}} \setminus T^1$  onto  $\hat{\mathbb{C}} \setminus V^2$  such that  $\psi_1(\psi_0^{-1}(y_0)) = y_0$  and  $\psi_0 \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_1$  on  $\hat{\mathbb{C}} \setminus T^1$ :

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus T^1 & \xrightarrow{\psi_1} & \hat{\mathbb{C}} \setminus T^2 \\ & \searrow \psi_0 \circ N_{p_1 e^{q_1}} & \downarrow N_{p_2 e^{q_2}} \\ & & \hat{\mathbb{C}} \setminus V^2. \end{array}$$

We extend  $\psi_1$  to the finite set  $T^1$  as a continuous map. Observe that  $\psi_1 = \psi_0 = \Psi$  on  $\Omega^0$ . The unique lift  $\tilde{\psi}_1$  of  $\psi_0^{-1} \circ N_{p_2 e^{q_2}}$  is a map from  $\hat{\mathbb{C}} \setminus T^2$  onto  $\hat{\mathbb{C}} \setminus V^1$  such that  $\tilde{\psi}_1(y_0) = \psi_0^{-1}(y_0)$  and  $\psi_0^{-1} \circ N_{p_2 e^{q_2}} = N_{p_1 e^{q_1}} \circ \tilde{\psi}_1$  on  $\hat{\mathbb{C}} \setminus T^2$ ;

$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus T^1 & \xleftarrow{\psi_1} & \hat{\mathbb{C}} \setminus T^2 \\ \downarrow N_{p_1 e^{q_1}} & \nearrow \psi_0^{-1} \circ N_{p_2 e^{q_2}} & \\ \hat{\mathbb{C}} \setminus V^1. & & \end{array}$$

Similarly, we extend  $\tilde{\psi}_1$  to the finite set  $T^2$  as a continuous map. It is easy to observe that  $\psi_1$  and  $\tilde{\psi}_1$  are inverses to each other on  $\hat{\mathbb{C}}$ . Moreover  $\psi_1$  is a quasiconformal homeomorphism with the same complex dilatation as  $\psi_0$ . By continuing this lifting process, as lifts are carried out with holomorphic maps we obtain a sequence of quasiconformal maps  $\{\psi_m\}_{m \geq 0}$  with the same bound on complex dilatation. Moreover, we have  $\psi_{m+1} = \psi_0 = \Psi$  on

$\Omega^0$  and  $\psi_m \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_{m+1}$  on  $\hat{\mathbb{C}}$ . Note that  $\psi_m = \psi_0$  is conformal on  $N_{p_1 e^{q_1}}^{-m}(\Omega^1)$ . The sequence  $\{\psi_m\}_{m \geq 0}$  is a normal family, so it has a converging sub-sequence, denoted it by  $\{\psi_{m_k}\}_{k \geq 0}$ ; and denote the limiting map by  $\psi_\infty$ . From the fact that the space of quasiconformal homeomorphisms with uniformly bounded dilatations is compact it follows that the homeomorphism  $\psi_\infty$  is quasiconformal. Note that, as constructed by lifts, the map  $\psi_\infty$  is conformal on  $\cup_{m=0}^\infty N_{p_1 e^{q_1}}^{-m}(\Omega^1)$ , the complement of which is a measure zero Julia set of  $N_{p_1 e^{q_1}}$ . Thus,  $\psi_\infty$  is conformal on  $\hat{\mathbb{C}}$ . We have  $\psi_\infty \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_\infty$  on  $\Omega^0$ .

Consider a rational function  $R = \psi_\infty^{-1} \circ N_{p_2 e^{q_2}} \circ \psi_\infty$ . Followed by the construction of the sequence of lifts, we have  $R = N_{p_1 e^{q_1}}$  on  $\Omega^0$ . By the identity principle of holomorphic functions we obtain  $R = N_{p_1 e^{q_1}}$  on  $\hat{\mathbb{C}}$ , i.e.  $\psi_\infty \circ N_{p_1 e^{q_1}} = N_{p_2 e^{q_2}} \circ \psi_\infty$  on  $\hat{\mathbb{C}}$ . The proof of surjectivity is finished here.  $\square$

## 6. PROOF OF MAIN THEOREM

Let us recall definitions of spaces we are working with. For a pair of natural numbers  $d \geq 3$  and  $1 \leq n \leq d$ , we have denoted by  $\mathcal{N}_{\text{pcm}}(d-n, n)$  the space of normalized postcritically minimal Newton maps  $N_{pe^q}$  of degree  $d \geq 3$  with  $n$  petals at  $\infty$ . It was denoted by  $\mathcal{N}_{\text{pcf}}^{+,n}(d)$  the space of normalized post-critically *finite* Newton maps with markings  $(\Delta_n^+)$ . Haïssinsky equivalence classes were denoted by  $\sim_H$ .

*Proof of Main Theorem 1.2.* For a pair of natural numbers  $d \geq 3$  and  $1 \leq n \leq d$ , define a map  $\mathcal{F}_n : \mathcal{N}_{\text{pcf}}^{+,n}(d) / \sim_H \rightarrow \mathcal{N}_{\text{pcm}}(d-n, n) / \mathbf{Affine}$  induced by parabolic surgery, i.e. for every postcritically minimal Newton map  $N_p$  with  $n$  marking  $\Delta_n^+$  apply Theorem 3.5, which results to a postcritically minimal Newton map  $N_{p_1 e^{q_1}}$  in  $\mathcal{N}_{\text{pcm}}(d-n, n)$ , normalize  $p_1$  and  $q_1$  if necessary. The mapping  $\mathcal{F}_n$  is well defined, indeed by Theorem 4.2 it follows that Haïssinsky equivalent classes of marked postcritically finite Newton maps produce affine conjugate results, moreover a (David) homeomorphism of the theorem preserves the dynamics and embedding of Julia sets. The mapping  $\mathcal{F}_n$  is also injective. Its surjectivity is followed by Theorem 5.1. We would like to remark that parabolic surgery is a natural bijection in a sense that the dynamics and embedding of Julia set are preserved. It is also unique in the sense that different choices of perturbations of a postcritically minimal Newton maps result to a unique postcritically finite Newton map of polynomial. Thus the correspondence is canonical.  $\square$

In [LMS], postcritically finite Newton maps of polynomials, of degree at least 3, have been classified in terms of connected finite graphs with certain properties. To get this finite data one needs to consider an iterated preimage of channel diagram, extended Hubbard tree of periodic superattracting cycles of period greater than 1 and Newton rays to connect the latter and its preimages to the former. If we include markings of channel diagram to

the data then by Main Theorem 1.2 a classification of postcritically minimal Newton maps of entire function, of degree at least 3, becomes an easy corollary of the classification of postcritically finite Newton maps.

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#### REFERENCES

- [BF14] B. Branner, N. Fagella, *Quasiconformal surgery in holomorphic dynamics*, Cambridge University Press, (2014).
- [BFJK] K. Barański, N. Fagella, X. Jarque, and B. Karpińska, *On the connectivity of the Julia sets of meromorphic functions*, Invent. Math., vol. 198, (2014), pp 591–636.
- [BCT14] X. Buff, G. Cui, and L. Tan. *Teichmüller spaces and holomorphic dynamics*. In Handbook of Teichmüller theory, volume IV, A. Papadopoulos, editor, European Mathematical Society, (2014), pp 717–756.
- [CG93] L. Carleson, T. Gamelin *Complex dynamics*, Universitext: Tracts in Mathematics. Springer-Verlag, New York, (1993).
- [Cil04] F. Cilingir, *On infinite area for complex exponential function*, Chaos Solitons Fractals, no. 5, vol. 22, (2004), pp 1189–1198.
- [CJ11] F. Cilingir, X. Jarque, *On Newton’s method applied to real polynomials*, J. Difference Equ. Appl., no. 6, 18, (2011), pp 1067–1076.
- [Cui] G. Cui, *Dynamics of rational maps: topology, deformation and bifurcation*, preprint 2009.
- [CT11] G. Cui, L. Tan, *A characterization of hyperbolic rational maps*, Invent. Math., no. 3, vol. 183, (2011), pp 451–516.
- [CT] G. Cui, L. Tan, *Hyperbolic-parabolic deformations of rational maps*, arXiv:1501.01385v3.
- [DH] A. Douady, J. H. Hubbard, *Exploring the Mandelbrot set. The Orsay Notes*.
- [DH93] A. Douady, J. H. Hubbard, *Proof of Thurston’s topological characterization of rational functions*, Acta math., vol. 171, (1993), pp 263–297.

- [Har99] M. Haruta, *Newtons method on the complex exponential function*, Trans. Amer. Math. Soc. no. 6, vol. 351, (1999), pp 2499–2513.
- [Ha98] P. Haïssinsky, *Chirurgie parabolique*, C. R. Math. Acad. Sci. Paris, vol. 327, (1998), pp 195–198.
- [LMS] R. Lodge, Y. Mikulich, and D. Schleicher, *A classification of post-critically finite Newton maps*. arXiv:1510.02771.
- [Ma15] K. Mamayusupov, *On Postcritically Minimal Newton maps*, PhD thesis, 2015.
- [Ma] K. Mamayusupov, *Post-critically minimal Newton maps of entire functions and parabolic surgery*, submitted, available at arXiv:1612.08643.
- [Man92] A. Manning, *How to be sure of finding a root of a complex polynomial using Newton’s method*, Bol. Soc. Bras. Mat vol. 22, (1992), pp 157–177.
- [MS06] S. Mayer, D. Schleicher *Immediate and virtual basins of Newton’s method for entire functions*, Ann. Inst. Fourier (Grenoble), no. 2, vol. 56, (2006), pp 325–336.
- [Mil06] J. Milnor, *Dynamics in one complex variable*, 3rd edition. Annals of Mathematics Studies, vol. 160. Princeton University Press, Princeton, NJ, (2006).
- [Prz89] F. Przytycki, *Remarks on the simple connectedness of basins of sinks for iterations of rational maps*, Dynamical Systems and Ergodic Theory, K. Krzyżewski (ed.), Banach Center Publications, Warsaw, Polish Sci. Publ., vol. 23, (1989), pp 229–235.
- [Shi09] M. Shishikura, *The connectivity of the Julia set of rational maps and fixed points*, In Complex Dynamics, Families and Friends: D. Schleicher, editor, A. K. Peters Ltd., Wellesley, MA, (2009), pp 257–276.
- [SS] D. Schleicher and R. Stoll, *Newton’s method in practice: finding all roots of polynomials of degree one million efficiently*, arXiv:1508.02935.
- [TY96] L. Tan, Y. Yongcheng, *Local connectivity of the Julia set for geometrically finite rational maps*, Science China mathematics, (1), vol. 39, (1996), pp 39–47.